WILD MULTIDEGREES OF THE FORM (d, d_2, d_3) FOR GIVEN d GREATHER THAN OR EQUAL TO 3

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Abstract. Let d be any number greather than or equal to 3. We show that the intersection of the set $\mathrm{mdeg}\,(\mathrm{Aut}\,(\mathbb{C}^3)) \backslash \mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^3))$ with $\{(d_1, d_2, d_3) \in (\mathbb{N}_+)^3 : d = d_1 \leq d_2 \leq d_3\}$ has infinitely many elements, where $\mathrm{mdeg}\, h = (\deg h_1, \ldots, \deg h_n)$ denotes the *multidegree* of a polynomial mapping $h = (h_1, \ldots, h_n) : \mathbb{C}^n \to \mathbb{C}^n$. In other words, we show that there is infiniltely many wild multidegrees of the form (d, d_2, d_3), with fixed $d \geq 3$ and $d \leq d_2 \leq d_3$, where a sequences $(d_1, \ldots, d_n) \in \mathbb{N}^n$ is a *wild multidegree* if there is a polynomial automorphism F of $\mathbb{C}^n$ with $\mathrm{mdeg}\, F = (d_1, \ldots, d_n)$, and there is no tame autmorphim of $\mathbb{C}^n$ with the same multidegree.

1. Introduction

In the following we will write $\mathrm{Aut}\,(\mathbb{C}^n)$ for the group of the all polynomial automorphisms of $\mathbb{C}^n$ and $\mathrm{Tame}\,(\mathbb{C}^n)$ for the subgroup of $\mathrm{Aut}\,(\mathbb{C}^n)$ containing all the tame automorphisms. Let us recall that a polynomial automorphism F is called *tame* if F can be expressed as a composition of linear and triangular automorphisms, where $G = (G_1, \ldots, G_n) \in \mathrm{Aut}\,(\mathbb{C}^n)$ is called *linear* if $\deg G_i = 1$ for $i = 1, \ldots n$, and $H = (H_1, \ldots, H_n) \in \mathrm{Aut}\,(\mathbb{C}^n)$ is called *triangular* if for some permutation σ of $\{1, \ldots, n\}$ we have $H_{\sigma(i)} - c_i \cdot x_{\sigma(i)}$ belongs to $\mathbb{C}[x_{\sigma(1)}, \ldots, x_{\sigma(i-1)}]$ for $i = 1, \ldots, n$ and some $c_i \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Here $\deg h$ denotes the total degree of a polynomial $h \in \mathbb{C}[x_1, \ldots, x_n]$.

Let $F = (f_1, \ldots, f_3) \in \mathrm{Aut}\,(\mathbb{C}^n)$. By *multidegree* of F we mean the sequence $\mathrm{mdeg}\, F = (\deg f_1, \ldots, \deg f_n)$. One can consider the function (also denoted mdeg) mapping $\mathrm{Aut}\,(\mathbb{C}^n))$ into $\mathbb{N}_+^n = (\mathbb{N} \setminus \{0\})^n$. It is well-known [1, 2] that

$$\mathrm{mdeg}\,(\mathrm{Aut}\,(\mathbb{C}^2)) = \mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^2)) = \{(d_1, d_2) \in \mathbb{N}_+^2 \,:\, d_1 | d_2 \text{ or } d_2 | d_1\}, \tag{1}$$

but in the higher dimension (even for $n = 3$) the situation is much more complicated and the question about the sets $\mathrm{mdeg}\,(\mathrm{Aut}\,(\mathbb{C}^n)$ and $\mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^n))$ is still not well recognized. The very first results [4] about the sets $\mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^n))$ for $n > 2$, say that $(3, 4, 5) \notin \mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^3))$ and $(d_1, \ldots, d_n) \in \mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^n))$ for all $d_1 \leq d_2 \leq \ldots \leq d_n$ with $d_1 \leq n-1$. Next, in [5] it was proved that for any prime numbers $p_2 > p_1 \geq 3$ and $d_3 \geq p_2$, we have $(p_1, p_2, d_3) \in \mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^3))$ if and only if $d_3 \in p_1\mathbb{N} + p_2\mathbb{N}$. The complete characterization of the set $\mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^3)) \cap \{(3, d_2, d_3) \,:\, 3 \leq d_2 \leq d_3\}$ was given in [6]. The result says that $(3, d_2, d_3)$, with $3 \leq d_2 \leq d_3$, belongs to $\mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^3))$ if and only if $3 | d_2$ or $d_3 \in 3\mathbb{N} + d_2\mathbb{N}$. The similar result about the set $\mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^3)) \cap \{(5, d_2, d_3) \,:\, 5 \leq d_2 \leq d_3\}$ and more other results are given in [7].

In the rest of the paper we will work with $n = 3$ and we will write $\mathbb{C}[x, y, z]$ instead of $\mathbb{C}[x_1, x_2, x_3]$. Let

$$\mathcal{W} = \mathrm{mdeg}\,(\mathrm{Aut}\,(\mathbb{C}^3))) \setminus \mathrm{mdeg}(\mathrm{Tame}\,(\mathbb{C}^3)) \tag{2}$$

and

$$\mathcal{W}_d = \mathcal{W} \cap \{(d_1, d_2, d_3) \in \mathbb{N}_+^3 : d = d_1 \leq d_2 \leq d_3\}. \tag{3}$$

Note that for the famous Nagata automorphism

$$N : \mathbb{C}^3 \ni \left\{ \begin{array}{c} x \\ y \\ z \end{array} \right\} \mapsto \left\{ \begin{array}{l} x - 2y(y^2 + zx) - z(y^2 + zx)^2 \\ y + z(y^2 + zx) \\ z \end{array} \right\} \in \mathbb{C}^3, \tag{4}$$

which is known to be wild automorphism, i.e. $N \notin \text{Tame}(\mathbb{C}^3)$, we have $\text{mdeg}\, N = (5,3,1) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$. Thus, $\text{mdeg}\, N$ is not an element of $\mathcal{W}$ (in other words, $\text{mdeg}\, N$ is not a wild multidegree). Besides of this the autors proved that the set $\mathcal{W}$ is not empty, and even more that this set is infinite [8]. Now we show the following refinement of that result:

Theorem 1.1. *Let $d > 2$ be any number. The set*

$$\mathcal{W}_d = \left[mdeg\,(Aut\,(\mathbb{C}^3)) \backslash mdeg\,(Tame\,(\mathbb{C}^3)) \right] \cap \left\{ (d_1, d_2, d_3) \in (\mathbb{N}_+)^3 : d = d_1 \leq d_2 \leq d_3 \right\}$$

is infinite.

The proof of the theorem will be given separetly for odd numbers $d \geq 3$ (section 2), even numbers $d > 4$ (section 3) and finally for $d = 4$ (section 4).

Note also the following remarks:

Remark 1.2. *The sets $\mathcal{W}_1$ and $\mathcal{W}_2$ are empty, i.e. if $d \in \{1, 2\}$ then for every $d_2, d_3 \in \mathbb{N}_+$ such that $d \leq d_2 \leq d_3$ one can show a tame automorphism F of $\mathbb{C}^3$ satisfying $mdeg\, F = (d, d_2, d_3)$.*

For $d = 1$ one can take $F(x,y,z) = (x, y + x^{d_2}, z + x^{d_3})$, while for $d = 2$ one can use [7, Cor. 3.3] or [4, Cor. 2.3].

Remark 1.3. *Let $d \leq e$ and define $\mathcal{W}_{d,e} = \{(d_1, d_2, d_3) \in \mathbb{N}_+^3 : d = d_1, e = d_2 \leq d_3\}$. Then the set $\mathcal{W}_{d,e}$ is finite.*

The proof of the above result can be found in [11] or [7, Thm. 8.1].

2. The case of odd number d

2.1. Elements of $\textbf{mdeg}\,(\textbf{Aut}\,(\mathbb{C}^3))$.

In this section we show the following two lemmas.

Lemma 2.1. *Let $r, k \in \mathbb{N}_+$. If $r \equiv 1 (mod\ 4)$, then*

$$(r, r + 2k, r + 4k) \in mdeg\,(Aut\,(\mathbb{C}^3)). \tag{5}$$

Proof. Since $r \equiv 1 (\text{mod } 4)$, we have $r = 4l + 1$ for some $l \in \mathbb{N}_+$. Let

$$F = (T \circ N_k) \circ (T \circ N_l), \tag{6}$$

where $T(x,y,z) = (z,y,x)$ and for any $m \in \mathbb{N}^*$

$$N_m : \mathbb{C}^3 \ni \left\{ \begin{array}{l} x \\ y \\ z \end{array} \right\} \mapsto \left\{ \begin{array}{l} x - 2y(y^2+zx)^m - z(y^2+zx)^{2m} \\ y + z(y^2+zx)^m \\ z \end{array} \right\} \in \mathbb{C}^n. \tag{7}$$

One can see that $\text{mdeg}\,(T \circ N_l) = (1, 1 + 2l, 1 + 4l)$. Moreover, if we put $(f,g,h) := T \circ N_l$, then $g^2 + fh = Y^2 + ZX$. Thus

$$\begin{aligned} F &= (T \circ N_k) \circ (f,g,h) \\ &= \left(h, g + h(Y^2+ZX)^k, f - 2g(Y^2+ZX)^k - h(Y^2+ZX)^{2k} \right). \end{aligned}$$

Since $\deg h > \max\{\deg f, \deg g\}$, one can see that

$$\text{mdeg}\, F = (4l + 1, (4l + 1) + 2k, (4l + 1) + 4k). \tag{8}$$

□

Lemma 2.2. *For every $r, k \in \mathbb{N}_+$, we have*

$$(r, r + k(r + 1), r + 2k(r + 1)) \in mdeg\,(Aut\,(\mathbb{C}^3)). \tag{9}$$

Proof. Assume that $r > 1$. Let

$$(f,g,h) = (X, Y, Z + X^r) \tag{10}$$

and put

$$F = (T \circ N_k) \circ (f,g,h). \tag{11}$$

Since

$$F = \big(h, g + h(g^2 + fh)^k, f - 2g(g^2 + fh)^k - z(g^2 + fh)^{2k}\big) \tag{12}$$

and $\deg h = r > \max\{\deg f, \deg g\}$, one can see that $\deg\big(g^2 + fh\big) = r + 1$ and so

$$\operatorname{mdeg} F = (r, r + k(r + 1), r + 2k(r + 1))\,. \tag{13}$$

If $r = 1$, then one can take $F = T \circ N_k$. $\square$

2.2. **Elements outside mdeg (Tame** $(\mathbb{C}^3)$**).** In this section we show the following two lemmas.

Lemma 2.3. *Let $r, k \in \mathbb{N}_+$. If $r > 1$ is odd and $\gcd(r, k) = 1$, then*

$$(r, r + 2k, r + 4k) \notin \mathit{mdeg}\,(\mathit{Tame}\,(\mathbb{C}^3)). \tag{14}$$

Lemma 2.4. *Let $r, k \in \mathbb{N}_+$. If $r > 1$ is odd and $\gcd(r, k) = 1$, then*

$$(r, r + k(r + 1), r + 2k(r + 1)) \notin \mathit{mdeg}\,(\mathit{Tame}\,(\mathbb{C}^3)). \tag{15}$$

In the proofs of the above lemmas we will use the following

Theorem 2.5 ([8], Thm. 2.1)**.** *Let $d_3 \geq d_2 > d_1 \geq 3$ be positive integers. If d_1 and d_2 are odd numbers such that $\gcd(d_1, d_2) = 1$, then $(d_1, d_2, d_3) \in \mathit{mdeg}\,(\mathit{Tame}\,(\mathbb{C}^3))$ if and only if $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$, i.e. if and only if d_3 is a linear combination of d_1 and d_2 with coefficients in $\mathbb{N}$.*

Proof of Lemma 2.3. Note that the numbers r and $r + 2k$ are odd. Moreover,

$$\gcd(r, r + 2k) = \gcd(r, 2k)\,, \tag{16}$$

and since r is odd,

$$\gcd(r, 2k) = \gcd(r, k) = 1. \tag{17}$$

Assume that $r + 4k \in r\mathbb{N} + (r + 2k)\mathbb{N}$. Since $2(r + 2k) > r + 4k$ and $r \nmid (r + 4k)$, we have

$$r + 4k = r + 2k + mr, \tag{18}$$

for some $m \in \mathbb{N}$. By (22), $2k = mr$. Since r is odd, the last equality means that $r|k$, a contradiction. Thus $r + 4k \notin r\mathbb{N} + (r + 2k)\mathbb{N}$, and by Theorem 2.5 we obtain a thesis. $\square$

Proof of Lemma 2.4. Since $r + 1$ is even, it follows that the numbers r and $r + k(r + 1)$ are odd. Moreover,

$$\gcd(r, r + k(r + 1)) = \gcd(r, k(r + 1))\,, \tag{19}$$

and since $\gcd(r, k) = 1$,

$$\gcd(r, k(r + 1)) = \gcd(r, r + 1) = \gcd(r, 1) = 1. \tag{20}$$

Similarily

$$\gcd(r, r + 2k(r + 1)) = \gcd(r, 2k(r + 1)) = \gcd(r, r + 1) = 1. \tag{21}$$

In particular $r \nmid r + 2k(r + 1)$.

Assume that $r+2k(r+1) \in r\mathbb{N}+(r + k(r + 1))\,\mathbb{N}$. Since $2(r+k(r+1)) > r+2k(r+1)$ and $r \nmid r+2k(r+1)$, we have

$$r + 2k(r + 1) = r + k(r + 1) + mr, \tag{22}$$

for some $m \in \mathbb{N}$. By (22), $k(r + 1) = mr$. Since $\gcd(r, k) = 1$, the last equality means that $r|r + 1$, a contradiction. Thus $r + 2k(r + 1) \notin r\mathbb{N} + (r + k(r + 1))\mathbb{N}$, and by Theorem 2.5 we obtain a thesis. $\square$

2.3. **Proof of the theorem in the case of odd d.** Take any odd number $d > 1$. If $d \equiv 1 (\mathrm{mod}\ 4)$, then by Lemmas 2.1 and 2.3 we have

$$\{(d, d + 2k, d + 4k) : \gcd(d, k) = 1\} \subset \mathrm{mdeg}\,(\mathrm{Aut}(\mathbb{C}^3)) \backslash \mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^3)). \tag{23}$$

If $d \equiv 3 (\mathrm{mod}\ 4)$, then by Lemmas 2.2 and 2.4 we have

$$\begin{aligned} &\{(d, d + k(d + 1), d + 2k(d + 1)) : \gcd(d, k) = 1\} \\ \subset\ & \mathrm{mdeg}\,(\mathrm{Aut}\,(\mathbb{C}^3)) \backslash \mathrm{mdeg}\,(\mathrm{Tame}\,(\mathbb{C}^3)). \end{aligned}$$

Since the set $\{k \in \mathbb{N}_+ : \gcd(d, k) = 1\}$ is infinite, the result follows.

3. The case of even number $d > 4$

3.1. **Preparatory calculations.** Fix even number $d > 4$ and take $k \in \mathbb{N}_+$ such that $\gcd(d, k) = 1$. Consider the automorphisms of $\mathbb{C}^3$:

$$H_d(x, y, z) = (x, y, z + x^d) \tag{24}$$

and N_k defined as in (7). Note that $N_k = \exp(D \cdot \sigma^k)$, where $D = \frac{\partial}{\partial z} + z\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial x}$ and $\sigma = y^2 + xz$. One can easily check that D is locally nilpotent derivation on $\mathbb{C}[x, y, z]$ and $\sigma \in \ker D$, so $\sigma^k \cdot D$ is also locally nilpotent. We will consider automorphisms $F_{d,k}$ of the form:

$$F_{d,k} = T \circ N_k \circ H_d \tag{25}$$

where T is defined as in the proof of Lemma 2.1. An easy calculation shows (even for $d = 4$) that

$$\mathrm{mdeg}\, F_{d,k} = (d, d + k(d + 1), d + 2k(d + 1)) \tag{26}$$

and writing $d_1 = d$, $d_2 = d + k(d + 1)$ and $d_3 = d + 2k(d + 1)$ gives

$$\gcd(d_1, d_2) = \gcd(d, d + k(d + 1)) = \gcd(d, k) = 1 \tag{27}$$

$$\gcd(d_2, d_3) = \gcd(d + k(d + 1), d + 2k(d + 1)) = \gcd(d + k(d + 1), d) = 1 \tag{28}$$

and

$$\gcd(d_1, d_3) = \gcd(d, d + 2k(d + 1)) = \gcd(d, 2k) = \gcd(d, 2) = 2. \tag{29}$$

We will prove that no tame automorphism of $\mathbb{C}^3$ has the same multidegree as $F_{d,k}$. Suppose to the contrary that $F = (F_1, F_2, F_3) \in \mathrm{Tame}(\mathbb{C}^3)$ and $\mathrm{mdeg}\, F = (d_1, d_2, d_3)$. As F is not linear, due to the result of Shestakov and Umirbaev [9, 10], F must admit an elementary reduction or a reduction of types I-IV (see e.g. [9, Def. 1-3]).

3.2. **Elementary reductions.** Recall that an elementary reduction on i-th coordinate F_i of F occurs when there exists $G(x, y) \in \mathbb{C}[x, y]$ such that $\deg(F_i - G(F_j, F_k)) < \deg F_i$, where $\{i, j, k\} = \{1, 2, 3\}$. We will use extensively the following

Proposition 3.1 (see e.g. [7, Prop. 2.7] or [9, Thm.2])**.** *Suppose that $f, g \in \mathbb{C}[X_1, \ldots, X_n]$ are algebraically independent and such that $\bar{f} \notin \mathbb{C}[\bar{g}]$ and $\bar{g} \notin \mathbb{C}[\bar{f}]$ ($\bar{h}$ denotes the highest homogeneous part of h). Assume that* $\deg f < \deg g$*, put*

$$p = \frac{\deg f}{\gcd(\deg f, \deg g)} \tag{30}$$

and suppose that $G(x, y) \in \mathbb{C}[x, y]$ with $\deg_y G(x, y) = pq + r$, $0 \le r < p$. Then

$$\deg G(f, g) \ge q(p \deg g - \deg g - \deg f + \deg[f, g]) + r \deg g \tag{31}$$

Suppose that F admits an elementary reduction on first coordinate, i.e. $\deg(F_1 - G(F_2, F_3)) < \deg F_1 = d_1$ for some $G \in \mathbb{C}[x, y]$. Consequently $\deg G(F_2, F_3) = d_1$. By (28), we know that $p := \frac{d_2}{\gcd(d_2, d_3)} = d_2$. Thus, from the above proposition applied to $f = F_2$ and $g = F_3$ we get

$$\deg G(F_2, F_3) \ge q(pd_3 - d_3 - d_2 + \deg[F_2, F_3]) + rd_3 \ge q(d_2 - 1)(d_3 - 1) + rd_3 \tag{32}$$

Since $d_1 < (d_2 - 1)(d_3 - 1)$ and $d_1 < d_3$ we obtain that $q = 0$ and $r = 0$. That is $\deg_y G(x, y) = 0$ and $G(x, y) = u(x)$. But then $d_1 = \deg G(F_2, F_3) = \deg u(F_2) = d_2 \cdot \deg u$, which is a contradiction.

Similarly, suppose that F admits an elementary reduction on third coordinate, i.e. $\deg(F_3-G(F_1,F_2)) < \deg F_3 = d_3$ for some $G \in \mathbb{C}[x,y]$. So $\deg G(F_1,F_2) = d_3$. Since $p := \frac{d_1}{\gcd(d_1,d_2)} = d \geq 3$ by (27), it follows that applying Proposition 3.1 to $f = F_1$ and $g = F_2$ we get

$$\deg G(F_1,F_2) \geq q(pd_2 - d_2 - d_1 + \deg[F_1,F_2]) + rd_2 \geq q(2k(d+1)+d+2) + rd_2. \tag{33}$$

Now, since $d_3 < 2k(d+1)+d+2$ and $d_3 < 2d_2$, we obtain that $q = 0$ and $r \in \{0,1\}$. If $r = 0$, we get $\deg_y G(x,y) = 0$ and so $G(x,y) = u(x)$. But then $d_3 = \deg G(F_1,F_2) = \deg u(F_1) = d_1 \cdot \deg u$, which is a contradiction because $\gcd(d_3,d_1) \leq 2 < d_1$. If $r = 1$, we get $G(x,y) = u(x) + yv(x)$ and so $d_3 = \deg G(F_1,F_2) = \deg(u(F_1) + F_2 v(F_1))$. Since $\deg F_1$ and $\deg F_2$ are coprime, $\deg(u(F_1) + F_2 v(F_1))$ must be equal either to $d_1 \cdot \deg u$ or to $d_2 + d_1 \cdot \deg v$. Consequently, $d_3 = d_1 \cdot \deg u$ or $d_3 = d_2 + d_1 \cdot \deg v$. First case leads to a contradiction since $\gcd(d_3,d_1) = 2 < d_1$ and second since $\gcd(d_3-d_2,d_1) = \gcd(k(d+1),d) = 1 < d_1$.

Now suppose that F admits an elementary reduction on second coordinate. Then $\deg(F_2-G(F_1,F_3)) < \deg F_2 = d_2$ for some $G \in \mathbb{C}[x,y]$ and so $\deg G(F_1,F_3) = d_2$. Let us put $p = \frac{d_1}{\gcd(d_1,d_3)}$ and apply Proposition 3.1 to $f = F_1$ and $g = F_3$. We will show that $\deg_y G(x,y) = 0$. By (29), $p = \frac{d}{2} \geq 2$ and so

$$\begin{aligned}
\deg G(F_1,F_3) &\geq q(pd_3 - d_3 - d_1 + \deg[F_1,F_3]) + rd_3 \\
&\geq q(\frac{d-2}{2}(2k(d+1)+d) - d + 2) + rd_3 \\
&\geq q((d-2)k(d+1)+2) + rd_3 \geq q(k(d+1)+d+2) + rd_3
\end{aligned}$$

Since $d_2 < k(d+1)+d+2$ and $d_2 < d_3$, we obtain that $q = 0$ and $r = 0$ so $\deg_y G(x,y) = 0$. Consequently, we get $G(x,y) = u(x)$. But then $d_2 = \deg G(F_1,F_3) = \deg u(F_1) = d_1 \cdot \deg u$, which is a contradiction since $\gcd(d_2,d_1) = 1 < d_1$.

To summarize: if F is a tame automorphism with multidegree equal to $\operatorname{mdeg} F_{d,k}$, then F does not admit an elementary reduction.

3.3. **Shestakov-Umirbaev reductions.** By the previous subsection and the following theorem we only need to check that no autmorphism of $\mathbb{C}^3$ with multidegree $(d_1,d_2,d_3) = (d, d+k(d+1), d+2k(d+1))$ admits a reduction of type III.

Theorem 3.2 ([7, Thm. 3.15])**.** *Let $(d_1,d_2,d_3) \neq (1,1,1)$, $d_1 \leq d_2 \leq d_3$, be a sequence of positive integers. To prove that there is no tame automorphism F of $\mathbb{C}^3$ with mdeg $F = (d_1,d_2,d_3)$ it is enough to show that a (hypothetical) automorphism F of $\mathbb{C}^3$ with mdeg $F = (d_1,d_2,d_3)$ admits neither a reduction of type III nor an elementary reduction. Moreover, if we additionally assume that $\frac{d_3}{d_2} = \frac{3}{2}$ or $3 \nmid d_1$, then it is enough to show that no (hypothetical) automorphism of $\mathbb{C}^3$ with multidegree (d_1,d_2,d_3) admits an elementary reduction. In both cases we can restrict our attention to automorphisms $F : \mathbb{C}^3 \to \mathbb{C}^3$ such that $F(0,0,0) = (0,0,0)$.*

But, since d_1 is even, it follows that $2 \nmid d_2$ by (27). Hence, no automorphism of $\mathbb{C}^3$ with multidegree (d_1,d_2,d_3) admits a reduction of type III by the following remark.

Remark 3.3 ([7, Rmk. 3.9])**.** *If an automorphism F of $\mathbb{C}^3$ with mdeg $F = (d_1,d_2,d_3)$, $1 \leq d_1 \leq d_2 \leq d_3$, admits a reduction of type III, then*
(1) $2|d_2$,
(2) $3|d_1$ *or* $\frac{d_3}{d_2} = \frac{3}{2}$.

4. The case of $d = 4$

Let us consider the mapping $F_{4,k}$ defined as in (25) for $k \in \mathbb{N}_+$ with $\gcd(4,k) = 1$ (in other words, for odd k). By (27) and (29), we know that d_2 is odd and d_3 is even. Then, since $d_3 - d_2 = 5k > 1$ and $\operatorname{mdeg} F_{4,k} = (4, 4+5k, 4+10k) =: (d_1,d_2,d_3)$ by (26), it follows that the result of Theorem 1.1, for $d = 4$, is a consequence of the following

Theorem 4.1 ([7, Thm. 6.10])**.** *If $d_2 \geq 5$ is odd and $d_3 \geq d_2$ is even such that $d_3 - d_2 \neq 1$, then $(4,d_2,d_3) \in$ mdeg $\left(\mathit{Tame}\left(\mathbb{C}^3\right)\right)$ if and only if $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$.*

In fact, if we assume that $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$, then we get $d_3 = d_2 + 4m$ for some $m \in \mathbb{N}$, since $2d_2 > d_3$ and $4 \nmid d_3$. Hence, $5k = d_3 - d_2 = 4m$. Since k is odd, this is a contradiction.

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